# 1617-257 Mon Sep 19, Hour 4: Topology in $R^n$, cont.

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on board

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Riddle Along: Can you place uncountably many disjoint Y shapes in $\mathbb{R}^2$?

Read Along: Munkres sec. 3.

**Def** $B(x,\epsilon)=\{y\in X: d(x,y)<\epsilon\}$

**Def** $U$ open $\Leftrightarrow \forall x\in U\ \exists \epsilon>0\ B(x,\epsilon)\subset U$

**Thm 1a** 1. $\emptyset, X$ are open

2. $V_\alpha$ open $\Rightarrow \bigcup V_\alpha$ is open

3. $V_i$ open $i=1,\dots n \Rightarrow \bigcap_{i=1}^n V_i$ is open

**Thm 4a** $U\subset\mathbb{R}^n$ is $\|\cdot\|$-open iff it is $|\cdot|$-open.

I should have prepared better.

**Thm 1b** 1. $\emptyset, X$ are closed

2. $F_\alpha$ closed $\Rightarrow \bigcap F_\alpha$ is closed.

3. $F_i$ closed, $i=1..n \Rightarrow \bigcup F_i$ is closed.

**Thm 4b** $F\subset\mathbb{R}^n$ is $\|\cdot\|$-closed iff it is $|\cdot|$-closed.

**Thm 2** $Y\subset X$ a subspace of a metric space.

then $A\subset Y$ is open iff $\exists U\subset X$ open s.t. $A=Y\cap U$, and $B\subset Y$ is closed iff $\exists F\subset X$ closed s.t. $B=Y\cap F$.

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done line

**Def** Limit $x_0$ of $A\subset X$:

$$\forall\epsilon>0\ \left(U(x_0,\epsilon)\setminus\{x_0\}\right)\cap A\neq\emptyset$$

Equiv: every nbd of $x_0$ contains $\infty$-many elements of $A$.

Closure: $\bar{A}=A\cup\{\text{limit pts of } A\}$

**Thm** $A$ is closed $\Leftrightarrow A=\bar{A}$

Ex: 1. $\bar{A}$ is the "smallest" closed set containing $A$.
2. $\bar{A}$ is the intersection of all closed sets containing $A$.

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Suppose $X$ & $Y$ are metric, w/ metrics $d_X$ & $d_Y$

**Def** $f: X \to Y$ is cont. at $x_0 \in X$ if for every nbd $V$ of $f(x_0)$ [:= an open set containing $f(x_0)$] there is a nbd $U$ of $x_0$ s.t. $f(U) \subset V$

$\iff$

$$\forall \epsilon > 0 \ \exists \delta > 0 \quad d_X(x, x_0) < \delta \Rightarrow d_Y(f(x), f(x_0)) < \epsilon$$

**Def** $f: X \to Y$ is cont. means $\forall x_0 \in X$, $f$ is cont. at $x_0$.

**Thm** TFAE for $f: X \to Y$:

1. $f$ is continuous.
2. For every $V$ open in $Y$, $f^{-1}(V)$ is open in $X$
3. For every $F$ closed in $Y$, $f^{-1}(F)$ is closed in $X$
4. if $X = Y = \mathbb{R}$, $f$ is cont. in the 157 sense.

**Thm** 1. constant functions are continuous.

2. $I: X \to X$ is cont.
3. $f: X \to Y$ cont, $A \subset X \Rightarrow f|_A : A \to Y$ is cont.
4. $f: X \to Y$, $g: Y \to Z \Rightarrow f/\!/g = g \circ f$ is cont.
5. $f: X \to \mathbb{R}^n$ is $(f_1(x), f_2(x), \dots f_n(x))$.
   Then $f$ is cont $\iff \forall i \ f_i$ is cont.
6. $f, g: X \to \mathbb{R}$ cont $\Rightarrow f+g, f \cdot g, f-g, \frac{f}{g}$ (where defined) cont.
7. $\pi_i : \mathbb{R}^n \to \mathbb{R}$ is cont.

Example: $f: \mathbb{R} \to \mathbb{R}$ by $x \mapsto x^x$ is cont.

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Skipped (but required): $f: X \to Y$, $\lim_{x \to x_0} f(x)$

.... some theorems ....

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**Def** $\operatorname{int} A =$ union of all open sets contained $= \{x \in A : \exists \epsilon > 0 \ U(x, \epsilon) \subset A\}$

in A

$\operatorname{Ext} A = \operatorname{int} A^c =$ union of all open sets disjoint from $A$.

$\operatorname{Bd} A = X \setminus (\operatorname{int} A \cup \operatorname{ext} A)$

<u>Claim</u> $\operatorname{int} A = \overline{(X \setminus A)}$ , $\operatorname{Ext} A = X \setminus \overline{A}$

$\operatorname{Bd} A = \overline{A} \cap \overline{(X \setminus A)}$

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HW: Read the rest of section 3, about limits, interiors, exteriors.